Laws of Probability, Bayes' theorem, and the Central Limit Theorem

G. Jogesh Babu
Department of Statistics
Penn State University

Do Random phenomena exist in Nature?

- Is a coin flip random?
  Not really, given enough information. But modeling the outcome as random gives a parsimonious representation of reality.
- Which way will an electron spin? Is it random?
  We can't exclude the possibility of a new theory being invented someday that would explain the spin, but modeling it as random is good enough.

Randomness is not total unpredictability; we may quantify that which is unpredictable.

Do Random phenomena exist in Nature?

If Mathematics and Probability theory were as well understood several centuries ago as they are today but the planetary motion was not understood, perhaps people would have modeled the occurrence of a Solar eclipse as a random event and assigned a probability based on empirical occurrence.

Subsequently, someone would have revised the model, observing that a solar eclipse occurs only on a new moon day. After more time, the phenomenon would be completely understood and the model changed from a stochastic, or random, model to a deterministic one.

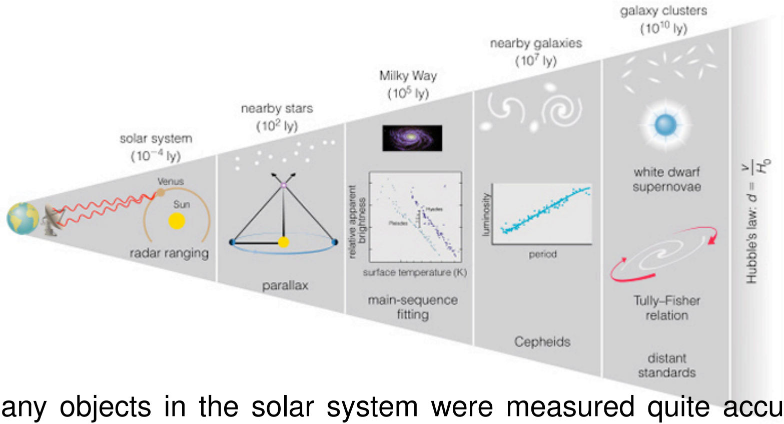
Do Random phenomena exist in Nature?

Thus, we often come across events whose outcome is uncertain. The uncertainty could be because of

- our inability to observe accurately all the inputs required to compute the outcome;
- excessive cost of observing all the inputs;
- lack of understanding of the phenomenon;
- dependence on choices to be made in the future, like the outcome of an election.

Cosmic distance ladder

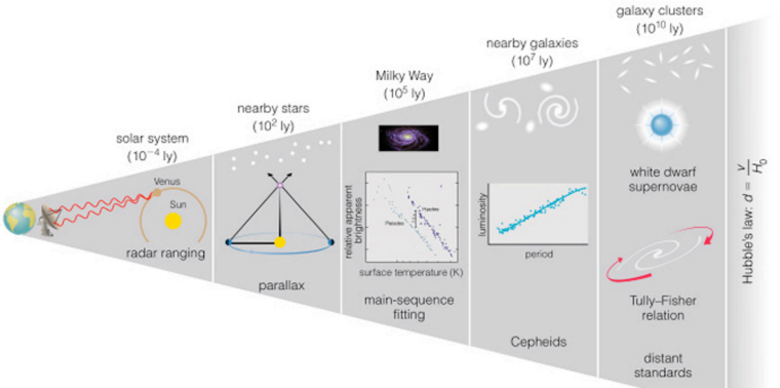
http://www.math.ucla.edu/~tao/preprints/Slides/Cosmic%20Distance%20Ladder.ppt



Many objects in the solar system were measured quite accurately by ancient Greeks and Babylonians using geometric and trigonometric methods.

Cosmic distance ladder

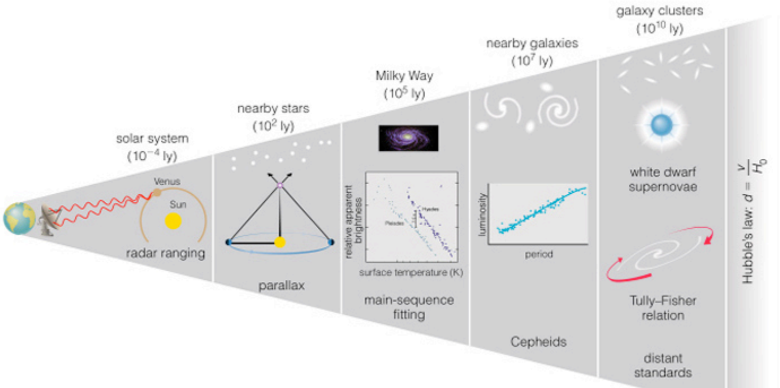
http://www.math.ucla.edu/~tao/preprints/Slides/Cosmic%20Distance%20Ladder.ppt



Distances to stars in the second rung are found by ideas of parallax, calculating the angular deviation over 6 months. First done by the mathematician Friedrich Bessel; accurate up to about 100 light years, though error is greater than on earlier rungs.

Cosmic distance ladder

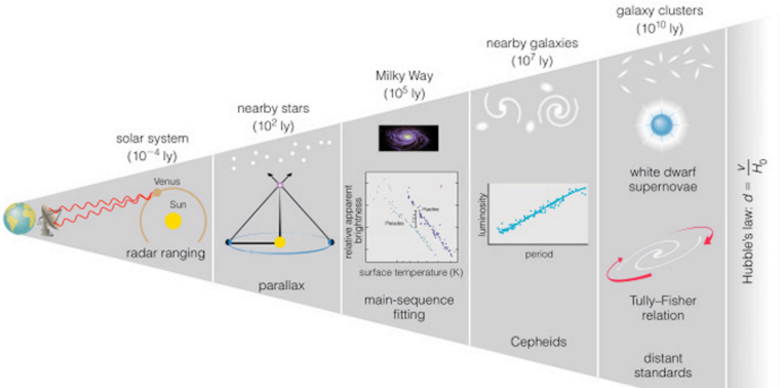
http://www.math.ucla.edu/~tao/preprints/Slides/Cosmic%20Distance%20Ladder.ppt



Distances of moderately far stars can be obtained by a combination of apparent brightness and distance to nearby stars using the Hertzsprung-Russell diagram. This method works for stars up to 300,000 light years and the error is significantly more.

Cosmic distance ladder

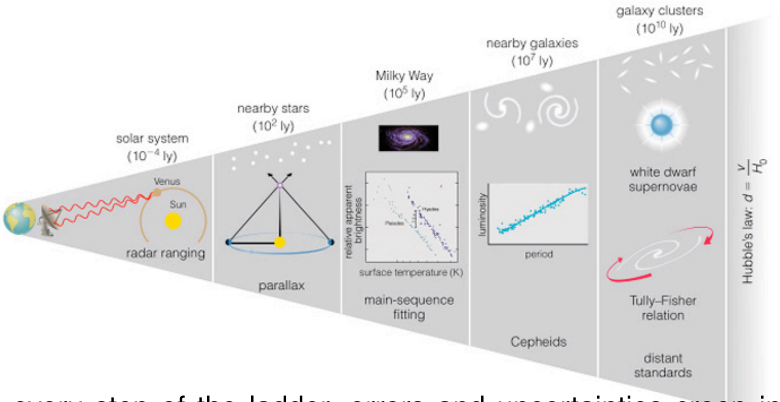
http://www.math.ucla.edu/~tao/preprints/Slides/Cosmic%20Distance%20Ladder.ppt



The distance to the next and final lot of stars is obtained by plotting the oscillations of their brightness. This method works for stars up to 13,000,000 light years away.

Cosmic distance ladder

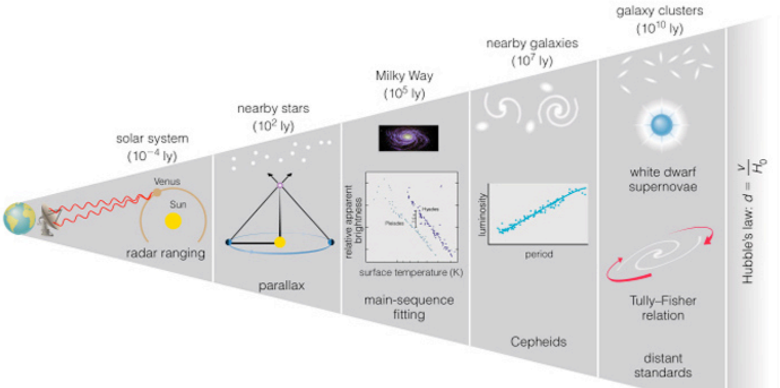
http://www.math.ucla.edu/~tao/preprints/Slides/Cosmic%20Distance%20Ladder.ppt



At every step of the ladder, errors and uncertainties creep in. Each step inherits all the problems of the ones below, and also the errors intrinsic to each step tend to get larger for the more distant objects.

Cosmic distance ladder

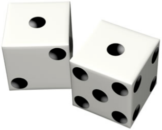
http://www.math.ucla.edu/~tao/preprints/Slides/Cosmic%20Distance%20Ladder.ppt



So we need to understand UNCERTAINTY. And one way of understanding a notion scientifically is to provide a structure to the notion. This structure must be rich enough to lend itself to quantification.

Coins etc.

The structure needed to understand a coin toss is intuitive. We assign a probability $1/2$ to the outcome HEAD and a probability $1/2$ to the outcome TAIL of appearing.



Similarly, for each of the outcomes 1,2,3,4,5,6 of the throw of a die, we assign a probability $1/6$ of appearing.

Similarly, for each of the outcomes $000001, \ldots, 999999$ of a lottery ticket, we assign a probability $1/999999$ of being the winning ticket.

Mathematical Formalization: Sample space

More generally, associated with any experiment we have a sample space Ω consisting of outcomes $\{o_1, o_2, \ldots, o_m\}$.

- Coin Toss: $\Omega = \{\text{H, T}\}$
- One die: $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Lottery: $\Omega = \{1, \ldots, 999999\}$

Each outcome is assigned a probability according to the physical understanding of the experiment.

- Coin Toss: $p_H = 1/2$, $p_T = 1/2$
- One die: $p_i = 1/6$ for $i = 1, \ldots, 6$
- Lottery: $p_i = 1/999999$ for $i = 1, \ldots, 999999$

Note that in each example, the sample space is *finite* and the probability assignment is *uniform* (i.e., the same for every outcome in the sample space), but this need not be the case.

Mathematical Formalization: Discrete Sample Space

- More generally, for an experiment with a *finite* sample space $\Omega = \{o_1, o_2, \ldots, o_m\}$, we assign a probability p_i to the outcome o_i for every i in such a way that the probabilities add up to 1, i.e., $p_1 + \cdots + p_m = 1$.
- In fact, the same holds for an experiment with a *countably infinite* sample space. (Example: Roll one die until you get your first six.)
- A finite or countably infinite sample space is sometimes called *discrete*.
- Uncountably infinite sample spaces exist, and there are some additional technical issues associated with them. We will hint at these issues without discussing them in detail.

Mathematical Formalization: Events

- A subset $E \subseteq \Omega$ is called an *event*.
- For a discrete sample space, this may if desired be taken as the mathematical definition of *event*.

Technical word of warning: If Ω is uncountably infinite, then we cannot in general allow arbitrary subsets to be called events; in strict mathematical terms, a *probability space* consists of:

- a sample space Ω,
- a set $\mathcal{F}$ of subsets of Ω that will be called the events,
- a function P that assigns a probability to each event and that must obey certain axioms.

Generally, we don't have to worry about these technical details in practice.

Mathematical Formalization: Random Variables

Back to the dice. Suppose we are gambling with one die and have a situation like this:

outcome	1	2	3	4	5	6
net dollars earned	-8	2	0	4	-2	4

Our interest in the outcome is only through its association with the monetary amount. So we are interested in a function from the outcome space Ω to the real numbers $\mathbb{R}$. Such a function is called a *random variable*.

Technical word of warning: A random variable must be a *measurable* function from Ω to $\mathbb{R}$, i.e., the inverse function applied to any interval subset of $\mathbb{R}$ must be an event in $\mathcal{F}$. For our discrete sample space, *any* map $X : \Omega \to \mathbb{R}$ works.

Mathematical Formalization: Random Variables

- Let X be the amount of money won on one throw of a die. We are interested in $\{X = x\} \subset \Omega$.

$\{X = 0\}$	for $x = 0$	Event $\{3\}$
$\{X = 2\}$	for $x = 2$	Event $\{2\}$
$\{X = 4\}$	for $x = 4$	Event $\{4, 6\}$
$\{X = -2\}$	for $x = -2$	Event $\{5\}$
$\{X = -8\}$	for $x = -8$	Event $\{1\}$
$\{X = 10\}$	for $x = 10$	Event $\emptyset$

- Notation is informative: Capital "X" is the random variable whereas lowercase "x" is some fixed value attainable by the random variable X.
- Thus X, Y, Z might stand for random variables, while x, y, z could denote specific points in the ranges of X, Y, and Z, respectively.

Mathematical Formalization: Random Variables

- Let X be the amount of money won on one throw of a die. We are interested in $\{X = x\} \subset \Omega$.

$$
\begin{array}{lll|l}
\{X=0\} & & \text{for } x=0 & \text{Event } \{3\} \\
\{X=2\} & & \text{for } x=2 & \text{Event } \{2\} \\
\{X=4\} & & \text{for } x=4 & \text{Event } \{4,6\} \\
\{X=-2\} & & \text{for } x=-2 & \text{Event } \{5\} \\
\{X=-8\} & & \text{for } x=-8 & \text{Event } \{1\} \\
\{X=10\} & & \text{for } x=10 & \text{Event } \emptyset
\end{array}
$$

- The probabilistic properties of a random variable are determined by the probabilities assigned to the outcomes of the underlying sample space.

Mathematical Formalization: Random Variables

- Let X be the amount of money won on one throw of a die. We are interested in $\{X = x\} \subset \Omega$.

$$
\begin{array}{lll|l}
\{X=0\} & & \text{for } x=0 & \text{Event } \{3\} \\
\{X=2\} & & \text{for } x=2 & \text{Event } \{2\} \\
\{X=4\} & & \text{for } x=4 & \text{Event } \{4,6\} \\
\{X=-2\} & & \text{for } x=-2 & \text{Event } \{5\} \\
\{X=-8\} & & \text{for } x=-8 & \text{Event } \{1\} \\
\{X=10\} & & \text{for } x=10 & \text{Event } \emptyset
\end{array}
$$

- Example: To find the probability that you win 4 dollars, i.e. $P(\{X = 4\})$, you want to find the probability assigned to the event $\{4, 6\}$. Thus

$$P\{\omega \in \Omega : X(\omega) = 4\} = P(\{4, 6\}) = (1/6) + (1/6) = 1/3.$$

Mathematical Formalization: Random Variables

$$P\{\omega \in \Omega : X(\omega) = 4\} = P(\{4,6\}) = 1/6 + 1/6 = 1/3.$$

- Adding $1/6 + 1/6$ to find $P(\{4,6\})$ uses a probability axiom known as *finite additivity*:

  Given disjoint events A and B, $P(A \cup B) = P(A) + P(B)$.

- In fact, any probability measure must satisfy *countable additivity*:

  Given mutually disjoint events $A_1, A_2, \ldots$, the probability of the (countably infinite) union equals the sum of the probabilities.

  N.B.: "Disjoint" means "having empty intersection".

Mathematical Formalization: Random Variables

$$P\{\omega \in \Omega : X(\omega) = 4\} = P(\{4, 6\}) = 1/6 + 1/6 = 1/3.$$

- $\{X = x\}$ is shorthand for $\{w \in \Omega : X(\omega) = x\}$
- If we summarize the possible nonzero values of $P(\{X = x\})$, we obtain a function of x called the *probability mass function* of X, sometimes denoted $f(x)$ or $p(x)$ or $f_X(x)$:

$$f_X(x) = P(\{X = x\}) = \begin{cases} 1/6 & \text{for } x = 0 \\ 1/6 & \text{for } x = 2 \\ 2/6 & \text{for } x = 4 \\ 1/6 & \text{for } x = -2 \\ 1/6 & \text{for } x = -8 \\ 0 & \text{for any other value of } x. \end{cases}$$

Mathematical Formalization: Probability Axioms

Any probability function P must satisfy these three axioms, where A and A_i denote arbitrary events:

- $P(A) \geq 0$ *(Nonnegativity)*
- If A is the whole sample space Ω then $P(A) = 1$
- If $A_1, A_2, \ldots$ are mutually exclusive (i.e., disjoint, which means that $A_i \cap A_j = \emptyset$ whenever $i \neq j$), then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \qquad \textit{(Countable additivity)}$$

Technical digression: If Ω is uncountably infinite, it turns out to be impossible to define P satisfying these axioms if "events" may be arbitrary subsets of Ω.

The Inclusion-Exclusion Rule

- For an event $A = \{o_{i_1}, o_{i_2}, \ldots, o_{i_k}\}$, we obtain

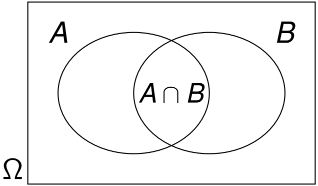
$$P(A) = p_{i_1} + p_{i_2} + \cdots + p_{i_k}.$$

- It is easy to check that if A, B are disjoint, i.e., $A \cap B = \emptyset$,

$$P(A \cup B) = P(A) + P(B).$$

- More generally, for *any* two events A and B,

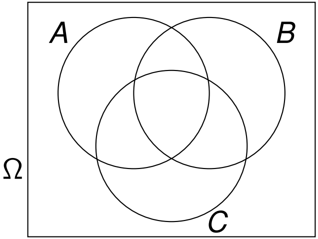
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$



The Inclusion-Exclusion Rule

- Similarly, for three events A, B, C

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\ &\quad -P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &\quad +P(A \cap B \cap C) \end{aligned}$$



- This identity has a generalization to n events called the *inclusion-exclusion rule*.

Assigning probabilities to outcomes

- Simplest case: Due to inherent symmetries, we can model each outcome in Ω as being equally likely.
- When Ω has m equally likely outcomes $o_1, o_2, \ldots, o_m$,

$$P(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{m}.$$

- **Well-known example:** If n people permute their n hats amongst themselves so that all $n!$ possible permutations are equally likely, what is the probability that at least one person gets his own hat?
  The answer,

$$1 - \sum_{i=0}^{n} \frac{(-1)^i}{i!} \approx 1 - \frac{1}{e} = 0.6321206\ldots,$$

  can be obtained using the inclusion-exclusion rule; try it yourself, then google "matching problem" if you get stuck.

Assigning probabilities to outcomes

Example: Toss a coin three times
Define X =Number of Heads in 3 tosses.

	$X(\Omega) = \{0, 1, 2, 3\}$
$\Omega = \{HHH,$	$\longleftarrow$ 3 Heads
$HHT, HTH, THH,$	$\longleftarrow$ 2 Heads
$HTT, THT, TTH,$	$\longleftarrow$ 1 Head
$TTT\}$	$\longleftarrow$ 0 Heads
$p(\{\omega\}) = 1/8$ for each $\omega \in \Omega$	$f(0) = 1/8,\ f(1) = 3/8,$ $f(2) = 3/8,\ f(3) = 1/8$

Conditional Probability

- Let X be the number that appears on the throw of a die.
- Each of the six outcomes is equally likely, but suppose I take a peek and tell you that X is an even number.
- Question: What is the probability that the outcome belongs to $\{1, 2, 3\}$?
- Given the information I conveyed, the six outcomes are no longer equally likely. Instead, the outcome is one of $\{2, 4, 6\}$ – each being equally likely.
- So *conditional on* the event $\{2, 4, 6\}$, the probability that the outcome belongs to $\{1, 2, 3\}$ equals 1/3.

More generally, consider an experiment with m equally likely outcomes and let A and B be two events. Given the information that B has occurred, the probability that A occurs is called the conditional probability of A given B and is written $P(A \mid B)$.

Conditional Probability

In general, when A and B are events such that $P(B) > 0$, the conditional probability of A given that B has occurred, $P(A \mid B)$, is defined by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Specific case: Uniform probabilities

Let $|A| = k$, $|B| = \ell$, $|A \cap B| = j$, $|\Omega| = m$. Given that B has happened, the new probability assignment gives a probability $1/\ell$ to each of the outcomes in B. Out of these ℓ outcomes of B, $|A \cap B| = j$ outcomes also belong to A. Hence

$$P(A \mid B) = j/\ell.$$

Noting that $P(A \cap B) = j/m$ and $P(B) = \ell/m$, it follows that

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Conditional Probability

- When A and B are events such that $P(B) > 0$, the conditional probability of A given that B has occurred, $P(A \mid B)$, is defined by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

- This leads to the *Multiplicative law of probability*,

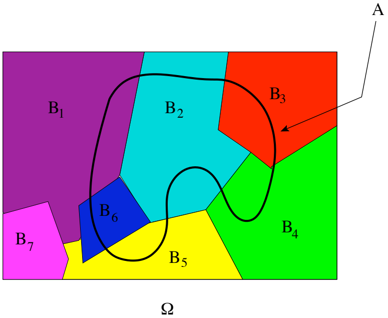
$$P(A \cap B) = P(A \mid B)P(B).$$

- This has a generalization to n events:

$$\begin{aligned} & P(A_1 \cap A_2 \cap \ldots \cap A_n) \\ = \quad & P(A_n \mid A_1, \ldots, A_{n-1}) \\ & \times P(A_{n-1} \mid A_1, \ldots, A_{n-2}) \\ & \times \ldots \times P(A_2 \mid A_1)P(A_1). \end{aligned}$$

The Law of Total Probability

Let $B_1, \ldots, B_k$ be a partition of the sample space Ω (a *partition* is a set of disjoint sets whose union is Ω), and let A be an arbitrary event:



Then

$$P(A) = P(A \cap B_1) + \cdots + P(A \cap B_k).$$

This is called the *Law of Total Probability*. Also, we know that $P(A \cap B_i) = P(A|B_i)P(B_i)$, so we obtain an alternative form of the Law of Total Probability:

$$P(A) = P(A|B_1)P(B_1) + \cdots + P(A|B_k)P(B_k).$$

The Law of Total Probability: An Example

Suppose a bag has 6 one-dollar coins, exactly one of which is a trick coin that has both sides HEADS. A coin is picked at random from the bag and this coin is tossed 4 times, and each toss yields HEADS.

Two questions which may be asked here are

- What is the probability of the occurrence of $A = \{\text{all four tosses yield HEADS}\}$?
- Given that A occurred, what is the probability that the coin picked was the trick coin?

The Law of Total Probability: An Example

- What is the probability of the occurrence of $A = \{\text{all four tosses yield HEADS}\}$?

This question may be answered by the Law of Total Probability. Define events

$$\begin{aligned} B &= \text{coin picked was a regular coin,} \\ B^c &= \text{coin picked was a trick coin.} \end{aligned}$$

Then B and B^c together form a partition of Ω. Therefore,

$$\begin{aligned} P(A) &= P(A \mid B)P(B) + P(A \mid B^c)P(B^c) \\ &= \left(\frac{1}{2}\right)^4 \times \frac{5}{6} + 1 \times \frac{1}{6} = \frac{7}{32}. \end{aligned}$$

Note: The fact that $P(A \mid B) = (1/2)^4$ utilizes the notion of independence, which we will cover shortly, but we may also obtain this fact using brute-force enumeration of the possible outcomes in four tosses if B is given.

The Law of Total Probability: An Example

- Given that A occurred, what is the probability that the coin picked was the trick coin?

For this question, we need to find

$$P(B^c \mid A) = \frac{P(B^c \cap A)}{P(A)} = \frac{P(A \mid B^c)P(B^c)}{P(A)}$$
$$= \frac{1 \times \frac{1}{6}}{\frac{7}{32}} = \frac{16}{21}.$$

Note that this makes sense: We should expect, after four straight heads, that the conditional probability of holding the trick coin, 16/21, is greater than the *prior* probability of 1/6 before we knew anything about the results of the four flips.

Bayes' Theorem

Suppose we have observed that A occurred.

- Let $B_1, \ldots, B_m$ be all possible scenarios under which A may occur, where $B_1, \ldots, B_m$ is a partition of the sample space.
- To quantify our suspicion that B_i was the cause for the occurrence of A, we would like to obtain $P(B_i \mid A)$.
- Here, we assume that finding $P(A \mid B_i)$ is straightforward for every i. (In statistical terms, a *model* for how A relies on B_i allows us to do this.)
- Furthermore, we assume that we have some prior notion of $P(B_i)$ for every i. (These probabilities are simply referred to collectively as our *prior*.)

Bayes' theorem is the prescription to obtain the quantity $P(B_i \mid A)$. It is the basis of Bayesian Inference. Simply put, our goal in finding $P(B_i \mid A)$ is to determine how our observation A modifies our probabilities of B_i.

Bayes' Theorem

Straightforward algebra reveals that

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{P(A)} = \frac{P(A \mid B_i)P(B_i)}{\sum_{j=1}^{m} P(A \mid B_j)P(B_j)}.$$

The above identity is what we call *Bayes' theorem*.

Thomas Bayes (?)

Note the apostrophe *after* the "s". The theorem is named for Thomas Bayes, an 18th-century British mathematician and Presbyterian minister.

The authenticity of the portrait shown here is a matter of some dispute.

Bayes' Theorem

Straightforward algebra reveals that

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{P(A)} = \frac{P(A \mid B_i)P(B_i)}{\sum_{j=1}^{m} P(A \mid B_j)P(B_j)}.$$

The above identity is what we call *Bayes' theorem*.

Observing that the denominator above does not depend on i, we may boil down Bayes' theorem to its essence:

$$\begin{aligned} P(B_i \mid A) &\propto P(A \mid B_i) \times P(B_i) \\ &= \text{“the likelihood (i.e., the model)”} \times \text{“the prior”}. \end{aligned}$$

Since we often call the left-hand side of the above equation the *posterior* probability of B_i, Bayes' theorem may be expressed succinctly by stating that the posterior is proportional to the likelihood times the prior.

Bayes' Theorem

Straightforward algebra reveals that

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{P(A)} = \frac{P(A \mid B_i)P(B_i)}{\sum_{j=1}^{m} P(A \mid B_j)P(B_j)}.$$

The above identity is what we call *Bayes' theorem*.

There are many controversies and apparent paradoxes associated with conditional probabilities. The root cause is sometimes incomplete specification of the conditions in a particular problem, though there are also some "paradoxes" that exploit people's seemingly inherent inability to modify prior probabilities correctly when faced with new information (particularly when those prior probabilities happen to be uniform).

Try googling "three card problem," "Monty Hall problem," or "Bertrand's box problem" if you're curious.

Independence

- Suppose that A and B are events such that

$$P(A \mid B) = P(A).$$

  In other words, the knowledge that B has occurred has not altered the probability of A.
- The Multiplicative Law of Probability tells us that in this case,

$$P(A \cap B) = P(A)P(B).$$

- When this latter equation holds, A and B are said to be *independent* events.

Note: The two equations here are not quite equivalent, since only the second is well-defined when B has probability zero. Thus, typically we take the second equation as the mathematical definition of independence.

Independence: More than two events

- It is tempting but not correct to attempt to define *mutual independence* of three or more events A, B, and C by requiring merely

$$P(A \cap B \cap C) = P(A)P(B)P(C).$$

  However, this equation does **not** imply that, e.g., A and B are independent.
- A sensible definition of mutual independence should include pairwise independence.
- Thus, we define mutual independence using a sort of recursive definition:

  > *A set of n events is mutually independent if the probability of its intersection equals the product of its probabilities* **and** *if all subsets of this set containing from 2 to $n-1$ elements are also mutually independent.*

Independence: Random variables

- Let X, Y, Z be random variables. Then X, Y, Z are said to be *independent* if

$$\begin{aligned} P(X \in S_1 \text{ and } Y \in S_2 \text{ and } Z \in S_3) \\ &= P(X \in S_1)P(Y \in S_2)P(Z \in S_3) \end{aligned}$$

  for **all** possible measurable subsets (S_1, S_2, S_3) of $\mathbb{R}$.
- This notion of independence can be generalized to any finite number of random variables (even two).
- Note the slight abuse of notation:

  "$P(X \in S_1)$" means "$P(\{\omega \in \Omega : X(\omega) \in S_1\})$".

Expectation of a Discrete Random Variable

- Let X be a random variable taking values $x_1, x_2 \ldots, x_n$. The expected value μ of X (also called the mean of X), denoted by $E(X)$, is defined by

$$\mu = E(X) = \sum_{i=1}^{n} x_i P(X = x_i).$$

Note: Sometimes physicists write $\langle X \rangle$ instead of $E(X)$, but we will use the more traditional statistical notation here.

- If $Y = g(X)$ for a real-valued function $g(\cdot)$, then, by the definition above,

$$E(Y) = E[g(X)] = \sum_{i=1}^{n} g(x_i) P(X = x_i).$$

Generally, we will simply write $E[g(X)]$ without defining an intermediate random variable $Y = g(X)$.

Variance of a Random Variable

- The variance σ^2 of a random variable is defined by

$$\sigma^2 = \mathrm{Var}(X) = E\left[(X-\mu)^2\right].$$

- Using the fact that the expectation operator is linear — i.e.,

$$E(aX + bY) = aE(X) + bE(Y)$$

  for any random variables X, Y and constants a, b — it is easy to show that

$$E\left[(X-\mu)^2\right] = E(X^2) - \mu^2.$$

- This latter form of $\mathrm{Var}(X)$ is usually easier to use for computational purposes.

Variance of a Random Variable

- Let X be a random variable taking values $+1$ or -1 with probability $1/2$ each.
- Let Y be a random variable taking values $+10$ or -10 with probability $1/2$ each.
- Then both X and Y have the same mean, namely 0, but a simple calculation shows that $\text{Var}(X) = 1$ and $\text{Var}(Y) = 100$.

This simple example illustrates that the variance of a random variable describes in some sense how spread apart the values taken by the random variable are.

Not All Random Variables Have An Expectation

As an example of a random variable with no expectation, suppose that X is defined on some (infinite) sample space Ω so that for all positive integers i,

$$X \text{ takes the value } \begin{cases} 2^i & \text{with probability } 2^{-i-1} \\ -2^i & \text{with probability } 2^{-i-1}. \end{cases}$$

Do you see why $E(X)$ cannot be defined in this example?

Both the positive part and the negative part of X have infinite expectation in this case, so $E(X)$ would have to be $\infty - \infty$, which is impossible to define.

The binomial random variable

- Consider n independent trials where the probability of "success" in each trial is $p \in (0, 1)$; let X denote the total number of successes.
- Then $P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$ for $x = 0, 1, \ldots n$.
- X is said to be a *binomial* random variable with parameters n and p, and this is written as $X \sim B(n, p)$.
- One may show that $E(X) = np$ and $Var(X) = np(1 - p)$.
- See, for example, A. Mészáros, "On the role of Bernoulli distribution in cosmology," *Astron. Astrophys.*, 328, 1-4 (1997). In this article, there are n uniformly distributed points in a region of volume $V = 1$ unit. Taking X to be the number of points in a fixed region of volume p, X has a binomial distribution. More specifically, $X \sim B(n, p)$.

The Poisson random variable

- Consider a random variable Y such that for some $\lambda > 0$,

$$P(Y = y) = \frac{\lambda^y}{y!} e^{-\lambda}$$

  for $y = 0, 1, 2, \ldots$.
- Then Y is said to be a *Poisson* random variable, written $Y \sim \text{Poisson}(\lambda)$.
- Here, one may show that $E(Y) = \lambda$ and $Var(Y) = \lambda$.
- If X has Binomial distribution $B(n, p)$ with large n and small p, then X can be approximated by a Poisson random variable Y with parameter $\lambda = np$, i.e.

$$P(X \leq a) \approx P(Y \leq a)$$

A Poisson random variable in the literature

See, for example, M. L. Fudge, T. D. Maclay, “Poisson validity for orbital debris ...” *Proc. SPIE*, 3116 (1997) 202-209.

The International Space Station is at risk from orbital debris and micrometeorite impact. How can one assess the risk of a micrometeorite impact?

A fundamental assumption underlying risk modeling is that orbital collision problem can be modeled using a Poisson distribution. “ ... assumption found to be appropriate based upon the Poisson ... as an approximation for the binomial distribution and ... that is it proper to physically model exposure to the orbital debris flux environment using the binomial distribution. ”

The Geometric Random Variable

- Consider n independent trials where the probability of "success" in each trial is $p \in (0, 1)$
- Unlike the binomial case in which the number of trials is fixed, let X denote the number of failures observed before the first success.
- Then
$$P(X = x) = (1 - p)^x p$$
for $x = 0, 1, \ldots$.
- X is said to be a *geometric* random variable with parameter p.
- Its expectation and variance are $E(X) = \frac{q}{p}$ and $Var(X) = \frac{q}{p^2}$, where $q = 1 - p$.

The Negative Binomial Random Variable

- Same setup as the geometric, but let X be the number of failures before observing r successes.
- $P(X = x) = \binom{r+x-1}{x} (1-p)^x p^r$ for $x = 0, 1, 2, \ldots$.
- X is said to be a *negative binomial* distributon with parameters r and p.
- Its expectation and variance are $E(X) = \frac{rq}{p}$ and $Var(X) = \frac{rq}{p^2}$, where $q = 1 - p$.
- The geometric distribution is a special case of the negative binomial distribution.
- See, for example, Neyman, Scott, and Shane (1953), On the Spatial Distribution of Galaxies: A specific model, *ApJ* **117**: 92–133. In this article, ν is the number of galaxies in a randomly chosen cluster. A basic assumption is that ν follows a negative binomial distribution.

Beyond Discreteness

- Earlier, we defined a *random variable* as a function from Ω to $\mathbb{R}$.
- For discrete Ω, this definition always works.
- But if Ω is uncountably infinite (e.g., if Ω is an interval in $\mathbb{R}$), we must be more careful:

  Definition: *A function $X : \Omega \to \mathbb{R}$ is said to be a* random variable *iff for all real numbers a, the set $\{\omega \in \Omega : X(\omega) \leq a\}$ is an event.*

- Fortunately, we can easily define “event” to be inclusive enough that the set of random variables is closed under all common operations.
- Thus, in practice we can basically ignore the technical details on this slide!

Distribution Functions and Density Functions

- The function F defined by

$$F(x) = P(X \leq x)$$

  is called the distribution function of X, or sometimes the cumulative distribution function, abbreviated c.d.f.

- If there exists a function f such that

$$F(x) = \int_{-\infty}^{x} f(t)dt \qquad \text{for all } x,$$

  then f is called a density of X.

- **Note:** The word "density" in probability is different from the word "density" in physics.

Distribution Functions and Density Functions

- The function F defined by

$$F(x) = P(X \leq x)$$

  is called the distribution function of X, or sometimes the cumulative distribution function, abbreviated c.d.f.

- If there exists a function f such that

$$F(x) = \int_{-\infty}^{x} f(t)dt \qquad \text{for all } x,$$

  then f is called a density of X.

- **Note:** It is typical to use capital "F" for the c.d.f. and lowercase "f" for the density function (recall that we earlier used f for the probability mass function; this creates no ambiguity because a random variable may not have both a density and a mass function).

Distribution Functions and Density Functions

- The function F defined by

$$F(x) = P(X \leq x)$$

is called the distribution function of X, or sometimes the cumulative distribution function, abbreviated c.d.f.

- If there exists a function f such that

$$F(x) = \int_{-\infty}^{x} f(t)dt \qquad \text{for all } x,$$

then f is called a density of X.

- **Note:** Every random variable has a uniquely defined c.d.f. $F(\cdot)$ and $F(x)$ is defined for all real numbers x.
In fact, $\lim_{x\to-\infty} F(x)$ and $\lim_{x\to\infty} F(x)$ always exist and are always equal to 0 and 1, respectively.

Distribution Functions and Density Functions

- The function F defined by

$$F(x) = P(X \leq x)$$

  is called the distribution function of X, or sometimes the cumulative distribution function, abbreviated c.d.f.
- If there exists a function f such that

$$F(x) = \int_{-\infty}^{x} f(t)dt \qquad \text{for all } x,$$

  then f is called a density of X.

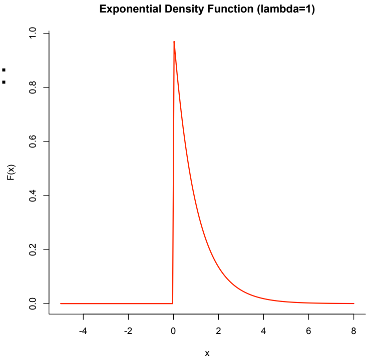
- **Note:** Sometimes a random variable X is called "continuous". This does not mean that $X(\omega)$ is a continuous function; rather, it means that $F(x)$ is a continuous function.
  Thus, it is technically preferable to say "X has a continuous distribution" instead of "X is a continuous random variable."

The Exponential Distribution

- Let $\lambda > 0$ be some positive parameter.
- The *exponential* distribution with mean $1/\lambda$ has desity

$$f(x) = \begin{cases} \lambda \exp(-\lambda x) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

The exponential density for $\lambda = 1$:

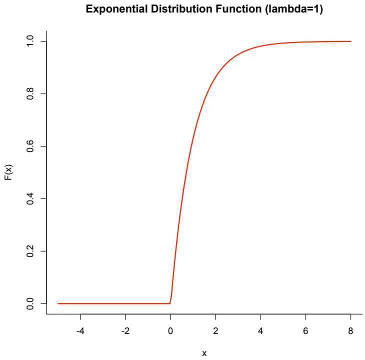


The Exponential Distribution

- Let $\lambda > 0$ be some positive parameter.
- The *exponential* distribution with mean $1/\lambda$ has c.d.f.

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 1 - \exp\{-\lambda x\} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The exponential c.d.f. for $\lambda = 1$:

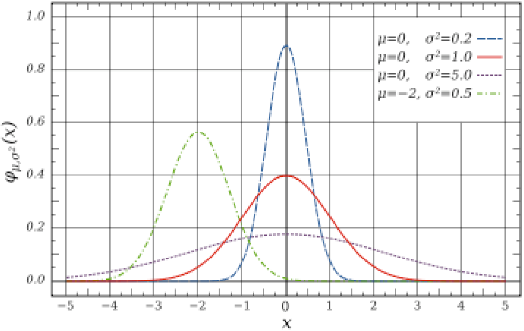


The Normal Distribution

- Let $\mu \in \mathbb{R}$ and $\sigma > 0$ be two parameters.
- The *normal* distribution with mean μ and variance σ^2 has desity

$$f(x) = \varphi_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}.$$

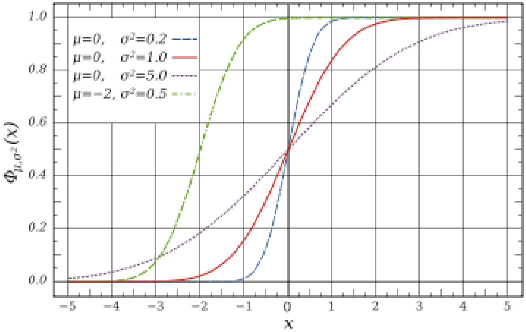
The normal density function for several values of (μ, σ^2):



The Normal Distribution

- Let $\mu \in \mathbb{R}$ and $\sigma > 0$ be two parameters.
- The *normal* distribution with mean μ and variance σ^2 has a c.d.f. without a closed form. But when $\mu = 0$ and $\sigma = 1$, the c.d.f. is sometimes denoted $\Phi(x)$.

The normal c.d.f. for several values of (μ, σ^2):



- Let $\mu \in \mathbb{R}$ and $\sigma > 0$ be two parameters.
- If $X \sim N(\mu, \sigma^2)$, then $\exp(X)$ has a *lognormal* distribution with parameters μ and σ^2.
- A common astronomical dataset that can be well-modeled by a shifted lognormal distribution is the set of luminosities of the globular clusters in a galaxy (technically, in this case the size of the shift would be a third parameter).
- The *lognormal* distribution with parameters μ and σ^2 has density

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\ln(x) - \mu)^2}{2\sigma^2}\right\} \text{ for } x > 0.$$

- With a shift equal to γ, the density becomes

$$f(x) = \frac{1}{(x-\gamma)\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\ln(x-\gamma) - \mu)^2}{2\sigma^2}\right\} \text{ for } x > \gamma.$$

Expectation for Continuous Distributions

- For a random variable X with density f, the expected value of $g(X)$, where g is a real-valued function defined on the range of X, is equal to

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx.$$

- Two common examples of this formula are given by the mean of X:

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

  and the variance of X:

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2.$$

Expectation for Continuous Distributions

- For a random variable X with normal density

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\},$$

  we have $E(X) = \mu$ and $Var(X) = E[(X-\mu)^2] = \sigma^2$.

- For a random variable Y with lognormal density

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}\right\} \text{ for } x > 0,$$

  We have

$$E(X) = e^{\mu+(\sigma^2/2)}$$

$$\mathrm{Var}(X) = E[(X-\mu)^2] = (e^{\sigma^2}-1)e^{2\mu+\sigma^2}.$$

Limit Theorems

- Define a random variable X on the sample space for some experiment such as a coin toss.
- When the experiment is conducted many times, we are generating a sequence of random variables.
- If the experiment never changes and the results of one experiment do not influence the results of any other, this sequence is called independent and identically distributed (i.i.d.).

- Suppose we gamble on the toss of a coin as follows: If HEADS appears then you give me 1 dollar and if TAILS appears then you give me -1 dollar, which means I give you 1 dollar.
- After n rounds of this game, we have generated an i.i.d. sequence of random variables $X_1, \ldots, X_n$, where each X_i satisfies
$$X_i = \begin{cases} +1 & \text{with prob. } 1/2 \\ -1 & \text{with prob. } 1/2. \end{cases}$$
- Then $S_n = X_1 + X_2 + \cdots + X_n$ represents my gain after playing n rounds of this game. We will discuss some of the properties of this S_n random variable.

- Recall: $S_n = X_1 + X_2 + \cdots + X_n$ represents my gain after playing n rounds of this game.
- Here are some possible events and their corresponding probabilities. Note that the proportion of games won is the same in each case.

OBSERVATION	PROBABILITY
$S_{10} \leq -2$ i.e. I lost at least 6 out of 10	0.38 moderate
$S_{100} \leq -20$ i.e. I lost at least 60 out of 100	0.03 unlikely
$S_{1000} \leq -200$ i.e. I lost at least 600 out of 1000	1.36×10^{-10} impossible

- Recall: $S_n = X_1 + X_2 + \cdots + X_n$ represents my gain after playing n rounds of this game.
- Here is a similar table:

OBSERVATION	PROPORTION	Probability
$\lvert S_{10}\rvert \leq 1$	$\frac{\lvert S_{10}\rvert}{10} \leq 0.1$	0.25
$\lvert S_{100}\rvert \leq 8$	$\frac{\lvert S_{100}\rvert}{100} \leq 0.08$	0.63
$\lvert S_{1000}\rvert \leq 40$	$\frac{\lvert S_{1000}\rvert}{1000} \leq 0.04$	0.81

- Notice the trend: As n increases, it appears that S_n is more likely to be near zero and less likely to be extreme-valued.

Law of Large Numbers

- Suppose $X_1, X_2, \ldots$ is a sequence of *i.i.d.* random variables with $E(X_1) = \mu < \infty$. Then

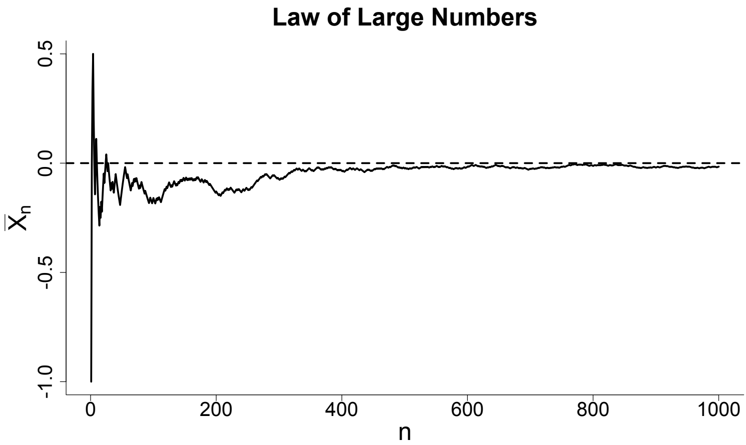
$$\overline{X}_n = \sum_{i=1}^{n} \frac{X_i}{n}$$

converges to $\mu = E(X_1)$ in the following sense: For any fixed $\epsilon > 0$,

$$P(\mid \overline{X}_n - \mu \mid > \epsilon) \longrightarrow 0 \text{ as } n \to \infty.$$

- In words: The sample mean $\overline{X}_n$ converges to the population mean μ.
- It is very important to understand the distinction between the sample mean, which is a random variable and depends on the data, and the true (population) mean, which is a constant.

In our example in which S_n is the sum of i.i.d. ± 1 variables, here is a plot of n vs. $\overline{X}_n = S_n/n$ for a simulation:



Central Limit Theorem

- Let $\Phi(x)$ denote the c.d.f. of a standard normal (mean 0, variance 1) distribution.
- Consider the following table, based on our earlier coin-flipping game:

Event	Probability	Normal
$S_{1000}/\sqrt{1000} \leq 0$	0.513	$\Phi(0) = 0.500$
$S_{1000}/\sqrt{1000} \leq 1$	0.852	$\Phi(1) = 0.841$
$S_{1000}/\sqrt{1000} \leq 1.64$	0.947	$\Phi(1.64) = 0.950$
$S_{1000}/\sqrt{1000} \leq 1.96$	0.973	$\Phi(1.96) = 0.975$

- It seems as though $S_{1000}/\sqrt{1000}$ behaves a bit like a standard normal random variable.

Central Limit Theorem

- Suppose $X_1, X_2, \ldots$ is a sequence of *i.i.d.* random variables such that $E(X_1^2) < \infty$.
- Let $\mu = E(X_1)$ and $\sigma^2 = E[(X_1 - \mu)^2]$. In our coin-flipping game, $\mu = 0$ and $\sigma^2 = 1$.
- Let

$$\overline{X}_n = \sum_{i=1}^{n} \frac{X_i}{n}.$$

- Remember: μ is the population mean and $\overline{X}_n$ is the sample mean.
- Then for any real x,

$$P\left\{\sqrt{n}\left(\frac{\overline{X}_n - \mu}{\sigma}\right) \leq x\right\} \longrightarrow \Phi(x) \text{ as } n \to \infty.$$

This fact is called the *Central Limit Theorem*.

Central Limit Theorem

The CLT is illustrated by the following figure, which gives histograms based on the coin-flipping game: